Exotic smoothness and particle physics.

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Abstract

Short introduction to exotic differential structures on manifolds is given. The possible physical context of this mathematical curiosity is discussed. The topic is very interesting although speculative.

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Classical differential calculus is defined on a Banach space. It has been generalized in two ways: the theory of generalized functions (distributions) and the calculus on (differential) manifolds. Both generalizations have found profound applications in physics. Here will discuss some aspects of the calculus on manifolds (differential geometry) [1-5]. Roughly speaking, a differential manifold is a topological space M that is locally homeomorphic to a Euclidean space (topological vector space in the general case). These local homeomorphisms form (provided they fulfil some consistency conditions on common domains) what we call an atlas on M. A real function $f : M \supset U_\alpha \rightarrow \mathbf{R}$ is said to be differentiable at $b \in U_\alpha$ if its local coordinate representation $f_\alpha = f \circ \phi_\alpha^{-1}$ is differentiable in the ordinary sense. The union of all consistent (that is $\phi_\alpha \circ \phi_\beta'^{-1}$ is a function of a given differentiability class) is called a differential structure on the manifold M. A function $f : M \rightarrow M'$ is said to be differentiable if its local coordinate representation $\phi_\beta \circ f \circ \phi_\alpha^{-1}$ is differentiable. If f^{-1} exists and is differentiable we call it difeomorphism and say that M and M'' are diffeomorphic (they are identical from the differential geometry point of view). One should ask the fundamental question *Can two homeomorphic manifolds (that is equivalent as topological spaces) support truly different (nondiffeomorphic) differential structures?* The answer is yes. What surprising is is the fact that $\mathbf{R^4}$, the four-dimensional Euclidean space, can be given infinitely many nondiffeomorphic (exotic) differential structures! In the following we will discuss possible physical consequences of this fact.

We will start by reviewing some aspects of exotic differential structures on $\mathbf{R^4}$ and other four-dimensional manifolds.

An exotic $\mathbf{R^4_\Theta}$ consists of a set of points which can be globally continuously identified with the set four coordinates (x^1, x^2, x^3, x^4). These coordinates may be smooth locally but they cannot be globally continued as smooth functions and no diffeomorphic image of an exotic $\mathbf{R^4_\Theta}$ can be given such global coordinates in a smooth way. There are uncountable many of different $\mathbf{R^4_\Theta}$. In fact, there is at least a two-parameter family of them [5]. C. H. Brans has proved the following theorem [7]:

Theorem 1. *There exist smooth manifolds which are homeomorphic but not diffeomorphic to* $\mathbf{R^4}$ *and for which the global coordinates* (t, x, y, z) *are smooth for* $x^2 + y^2 + z^2 \geq a^2 > 0$*, but not globally. Smooth metrics exist for which the boundary of this region is timelike, so that the exoticness is spatially confined.*

Of course, there are also $\mathbf{R^4_\Theta}$ whose exoticness cannot be localized. They might have important cosmological consequences. We also have [7]

Theorem 2. *If* M *is a smooth connected 4-manifolds and* S *is a closed submanifold for which* $H^4(M, S, \mathbf{Z}) = 0$*, then any smooth, time-orientable Lorentz metric defined over* S *can be smoothly continued to all of* M*.*

It can also be proven that if you remove one point from a four-manifold

then the resulting manifold has its exotic versions [5]. For example, by removing a point from $\mathbf{R^4}$ we obtain a manifold that is topologically $\mathbf{R} \times \mathbf{S^3}$ and has exotic differential structures that might be very important for cosmologists.

The discussion of possible physical consequences of the existence of exotic differential structures on some four-manifolds is very difficult because we lack such important "details" as explicit construction of a metric tensor and so on. Nevertheless, some general remarks can be given. Brans has even conjectured [7] that a localized exoticness (in the sense of the Theorem 1) can act as a source for some externally regular field just as matter can. To define a "reasonable" quantum field theory on a manifold we need a notion of "positive frequency" in the asymptotic past and future ("in" and "out" states). This is not an easy task for a general spacetime. R. Wald has told us [8] how to define a quantum theory in the case of curvature of compact support (the spacetime becomes flat in the past and future). This means that it might be possible to construct a quantum field theory (scattering matrix) on some exotic $R \times S^3$. Spacetime of this topology arises if we require that all physical fields vanish at spatial infinity. Cosmologists are also exploring such spacetime manifolds.

There is another reason for believing that exotic smoothess has a potential physical context. The 28 differential structures on the S^7 and some homeomorphic homogeneous spaces can be distinguished by their spectra

provided an appropriate metrics is chosen (the Pontrjagin forms must vanish). To be more precise we have [9]:

Collorary *Suppose M and M' are two topological k-spheres (with codimension one metrics), k= 7 or 11. If M and M' are isospectral then they are diffeomorphic.*

Kreck and Stolz have shown [10] that certain Einstein seven-manifolds with $SU(3)\times SU(2)\times U(1)$ symmetry are distinguished by their spectra [10]. They have also given an example of an Einstein manifold with an exotic structure admitting again an Einstein metrics. Stolz [11] has shown that exotic differential structures on some four-manifolds can be detected by spectral invariants of the twisted Dirac operator. For example, $\eta(\mathbf{RP^4}, g, \phi) \neq \eta(\mathbf{Q^4}, g', \phi')$ for all metrics g and g'. Here $\mathbf{Q^4}$ denotes an exotic version of the real four-dimensional projective space and ϕ the pin-structure. η is the famous eta invariant (asymmetry of the spectrum of the Dirac operator). All these examples are important from the Kaluza-Klein or string-inspired models because spectra of internal spaces often determine physical data [12, 13].

Let us now consider the A. Connes construction of the standard model lagrangian [14-17]. The spacetime consists of at least two copies of the ordinary spacetime manifold. One may ask if both manifolds carry the same differential structure. If not [18] we must impose some consistency conditions to make to make the fields defined with respect to different differential

structures compatible. The simplest and easiest condition to fulfil is to demand that field smooth with respect to one differential structure must have compact supports (constant function are smooth so there is no smoothness problem outside the support). This means that exotic smoothness may be a source of sort of confinement if the Connes construction is correct (bag-like structures). This might be important from the astrophysical/cosmological point of view (dark matter?).

Let me conclude this sketchy review by stating that exotic smoothness is interesting not only as a mathematical curiosity but also for its physical context. If Nature have not used exotic smoothness we should find why. We should also know if and why only one of the existing differential structures has been chosen. Does it mean that calculus, although very powerful is not necessary to describe physical phenomena? It might not be easy to find any answer to these questions.

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